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Quadratic reductions of quadrilateral lattices

Adam Doliwa *

Instituto Nazionale di Fisica Nucleare, Sezione di Roma P-le Aldo Moro 2, I-00185 Rome, Italy

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Abstract

It is shown that quadratic constraints are compatible with the geometric integrability scheme of the multidimensional quadrilateral lattice equation. The corresponding Ribaucour-type reduction of the fundamental transformation of quadrilateral lattices is found as well, and superposition of the Ribaucour transformations is presented in the vectorial framework. Finally, the quadratic reduction approach is illustrated on the example of multidimensional circular lattices. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The connection between differential geometry and modern theory of integrable partial differential equations has been observed many times [2,20,34,35]. Actually, a lot of basic integrable systems, like the sine-Gordon, Liouville, Lamé or Darboux equations, were studied by distinguished geometers of the XIXth century [1,9,26] (see also [21,27]).

It turns out that many integrable systems of a geometric origin are reductions of the Darboux equations, which describe submanifolds in $\mathbb{R}^M$ parameterized by conjugate coordinate systems (conjugate nets) [9]. The Darboux equations were rediscovered and solved, in the matrix generalization, in [37] using the $\bar{\partial}$-dressing method (for another approach to the Darboux equations see [22,23] and references therein). More recently it was shown in [19] that classical transformations of the conjugate nets, which are known as the Laplace,

* Permanent address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski ul. Hoża 69, 00-681 Warszawa, Poland.

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Lévy, Combescure, radial and fundamental transformations [21,27], provide an interesting geometric interpretation of the basic operations associated with the multicomponent Kadomtsev–Petviashvilii hierarchy.

The distinguished reduction of the Darboux equations is given by the Lamé equations [1,9,26], which describe orthogonal systems of coordinates; these equations were solved recently in [36]. The reduction of the fundamental transformation compatible with the orthogonality constraint is provided by the Ribaucour transformation [1], which vectorial generalization was constructed in [28].

During last few years the connection between geometry and integrability was observed also at a discrete level [4–6,15]. In particular, in [16] it was shown that the integrable discretization of the conjugate nets is provided by multidimensional quadrilateral lattices (MQL), i.e., maps $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^M$ with all the elementary quadrilaterals planar (see also [13,32]). The geometrically distinguished reduction of quadrilateral lattices are multidimensional circular lattices (MCL), for which all the elementary quadrilaterals should be inscribed in circles [3,8]. The circular lattices provide the integrable discretization of the orthogonal coordinate systems of Lamé. In [8,17] it was demonstrated that the circularity constraint is compatible with the geometric and analytic integrability scheme of MQL and provides an integrable reduction of the corresponding equations.

Also the Darboux-type transformations [30] of quadrilateral lattices have been studied from various points of view [13,18,25,29]. In [18] the general theory of transformations applicable to any quadrilateral lattice was presented, and all the classical transformations of conjugate nets have been generalized to a discrete level. In [25] there was found, among others, the (discrete analog of the) Ribaucour transformation compatible with the circularity constraint.

In conclusion of [8] there were given some arguments supporting the conjecture that a more general than circularity, but still *quadratic*, constraint imposed on the quadrilateral lattice preserves their integrability scheme. In the present paper we develop this observation and prove the general theorem about the integrability of quadratic reductions of the MQL equation (Section 2). The corresponding Ribaucour reduction of the fundamental (binary Darboux) transformation is found in Section 3, where we present also superposition of the Ribaucour transformations in the vectorial framework. Section 4 provides an exposition of the multidimensional circular lattices and their Ribaucour transformation from the quadratic reduction point of view.

2. Integrability of quadratic reductions

We recall that planarity of elementary quadrilaterals of the lattice $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^M$ can be expressed in terms of the Laplace equations [13,16]:

$$\Delta_i \Delta_j \boldsymbol{x} = (T_i A_{ij})\Delta_i \boldsymbol{x} + (T_j A_{ji})\Delta_j \boldsymbol{x}, \quad i \neq j, \quad i, j = 1, \dots, N, \tag{1}$$

where the coefficients A_{ij}, for $N > 2$, due to the compatibility condition of (1), satisfy the MQL equation [7,16]:

$$\Delta_k A_{ij} = (T_j A_{jk})A_{ij} + (T_k A_{kj})A_{ik} - (T_k A_{ij})A_{ik}, \quad i \neq j \neq k \neq i \tag{2}$$

in the above formulas T_i is the shift operator in the ith direction of the lattice and $\Delta_i = T_i - 1$ is the corresponding partial difference operator.

In this paper we study lattices $\boldsymbol{x}$ contained in a quadric hypersurface $\mathcal{Q}$ of $\mathbb{R}^M$, $N \leq M-1$. This additional constraint implies that the lattice points $\boldsymbol{x}$ satisfy the equation of the quadric $\mathcal{Q}$, which we write in the form

$$\boldsymbol{x}^{\mathrm{t}} Q \boldsymbol{x} + \boldsymbol{a}^{\mathrm{t}} \boldsymbol{x} + c = 0; \tag{3}$$

here Q is a symmetric matrix, $\boldsymbol{a}$ is a constant vector, c is a scalar and t denotes transpositon.

Let us recall (see [16] for details) that the construction scheme of a generic N-dimensional quadrilateral lattice ($N > 2$) involves the linear operations only, and is a consequence of the planarity of elementary quadrilaterals (or the Laplace equations (1)).

Theorem 1. *The point $T_i T_j T_k \boldsymbol{x}$ of the lattice is the intersection point of the three planes $\mathbb{V}_{jk}(T_i\boldsymbol{x}) = \langle T_i\boldsymbol{x}, T_iT_j\boldsymbol{x}, T_iT_k\boldsymbol{x}\rangle$, $\mathbb{V}_{ik}(T_j\boldsymbol{x}) = \langle T_j\boldsymbol{x}, T_iT_j\boldsymbol{x}, T_jT_k\boldsymbol{x}\rangle$ and $\mathbb{V}_{ij}(T_k\boldsymbol{x}) = \langle T_k\boldsymbol{x}, T_iT_k\boldsymbol{x}, T_jT_k\boldsymbol{x}\rangle$ in the three-dimensional space $\mathbb{V}_{ijk}(\boldsymbol{x}) = \langle \boldsymbol{x}, T_i\boldsymbol{x}, T_j\boldsymbol{x}, T_k\boldsymbol{x}\rangle$.*

Remark. *The above construction scheme is the geometrical counterpart of MQL equations (2) and it is called, therefore, the geometric integrability scheme. It implies, in particular, that the lattice $\boldsymbol{x}$ is completely determined once a system of initial quadrilateral surfaces has been given* [16].

As it was proposed in [8], a geometric constraint in order to be integrable must "propagate" in the construction of the MQL, when satisfied by the initial surfaces. The (geometric) integrability of the quadratic reductions is an immediate consequence of the following classical *eight points theorem* (see, for example [31, pp. 420, 424]).

Lemma 1. *Given eight distinct points which are the set of intersections of three quadric surfaces, all quadrics through any subset of seven of the points must pass through the eight point.*

Proposition 1. *Quadratic reductions of quadrilateral lattices are compatible with geometric integrability scheme of the multidimensional quadrilateral lattice equation.*

Proof. Since the construction of the MQL for arbitrary $N \geq 3$ can be reduced to the compatible construction of three-dimensional quadrilateral lattices [16], it is enough to show that the constraint is preserved in a single step described in Theorem 1. We must show that if the seven points $\boldsymbol{x}$, $T_i\boldsymbol{x}$, $T_j\boldsymbol{x}$, $T_k\boldsymbol{x}$, $T_iT_j\boldsymbol{x}$, $T_iT_k\boldsymbol{x}$ and $T_jT_k\boldsymbol{x}$ belong to the quadric $\mathcal{Q}$, then the same holds for the eight point $T_iT_jT_k\boldsymbol{x}$ as well. Denote by $\mathcal{Q}_{ijk}(\boldsymbol{x})$ the intersection of the quadric $\mathcal{Q}$ with the three-dimensional space $\mathbb{V}_{ijk}(\boldsymbol{x})$, there are two possibilities:

(i) $\mathcal{Q}_{ijk}(\boldsymbol{x}) = \mathbb{V}_{ijk}(\boldsymbol{x})$, or
(ii) $\mathcal{Q}_{ijk}(\boldsymbol{x}) \subset \mathbb{V}_{ijk}(\boldsymbol{x})$ is a quadric surface.

Since in the first case the conclusion is trivial, we concentrate on the second point. Recall that two planes in $\mathbb{V}_{ijk}$ can be considered as a degenerate quadric surface (in this case the quadratic equation splits into two linear factors). Application of Lemma 1 to three (degenerate) quadric surfaces $\mathbb{V}_{ij}(\boldsymbol{x}) \cup \mathbb{V}_{ij}(T_k\boldsymbol{x})$, $\mathbb{V}_{ik}(\boldsymbol{x}) \cup \mathbb{V}_{ik}(T_j\boldsymbol{x})$, $\mathbb{V}_{jk}(\boldsymbol{x}) \cup \mathbb{V}_{jk}(T_i\boldsymbol{x})$ and to the fourth one $\mathcal{Q}_{ijk}(\boldsymbol{x})$ concludes the proof. □

Corollary 1. *The above result can be obviously generalized to quadrilateral lattices in spaces obtained by intersection of many quadric hypersurfaces. Since the spaces of constant curvature, Grassmann manifolds and Segré or Veronese varieties can be realized in this way* [24], *the above results can be applied, in principle, to construct integrable lattices in such spaces as well.*

3. Ribaucour transformation

In this section we suitably adapt the fundamental transformation of quadri lateral lattices in order to preserve a given quadratic constraint. Such reductions are called, in the continuous context, the Ribaucour transformations (see, for example, [21, Chapter X]).

3.1. Ribaucour reduction of the fundamental transformation

We first recall (for details, see [18]) the basic results concerning the fundamental transformation of quadrilateral lattices.

Theorem 2. *The fundamental transformation $\mathcal{F}(\boldsymbol{x})$ of the quadrilateral lattice $\boldsymbol{x}$ is given by*

$$\mathcal{F}(\boldsymbol{x}) = \boldsymbol{x} - \frac{\phi}{\phi_{\mathrm{C}}}\boldsymbol{x}_{\mathrm{C}}, \tag{4}$$

where:

(i) $\phi : \mathbb{Z}^N \to \mathbb{R}$ *is a new solution of the Laplace equation* (1) *of the lattice* $\boldsymbol{x}$

(ii) $\boldsymbol{x}_{\mathrm{C}}$ *is the Combescure transformation vector, which is a solution of the equations*

$$\Delta_i \boldsymbol{x}_{\mathrm{C}} = (T_i\sigma_i)\Delta_i\boldsymbol{x}, \tag{5}$$

where, due to the compatibility of the system (5) *the functions* σ_i *satisfy*

$$\Delta_j\sigma_i = A_{ij}(T_j\sigma_j - T_j\sigma_i), \quad i \neq j; \tag{6}$$

moreover,

(iii) ϕ_{C} *is a solution, corresponding to* ϕ, *of the Laplace equation of the lattice* $\boldsymbol{x}_{\mathrm{C}}$, *i.e.,*

$$\Delta_i\phi_{\mathrm{C}} = (T_i\sigma_i)\Delta_i\phi. \tag{7}$$

Remark. *Notice that, given $\boldsymbol{x}_{\mathrm{C}}$ and ϕ, Eq.* (7) *determines ϕ_{C} uniquely, up to a constant of integration.*

At this point we also recall (see [18] for details) that an N parameter family of straight lines in $\mathbb{R}^M$ is called *N-dimensional congruence* if any two neighbouring lines $\mathfrak{l}$ and $T_i\mathfrak{l}$, $i = 1, \ldots, N$ of the family are coplanar. The N-dimensional quadrilateral lattice $\boldsymbol{x}$ and N-dimensional congruence are called *conjugate*, if $\boldsymbol{x}(\boldsymbol{n}) \in \mathfrak{l}(\boldsymbol{n})$, for every $\boldsymbol{n} \in \mathbb{Z}^N$.

Corollary 2. *The N parameter family of lines $\mathfrak{l} = \langle \boldsymbol{x}, \mathcal{F}(\boldsymbol{x})\rangle$ forms a congruence, called congruence of the transformation. Both lattices $\boldsymbol{x}$ and $\mathcal{F}(\boldsymbol{x})$ are conjugate to the congruence $\mathfrak{l}$.*

Theorem 2 states that in order to construct the fundamental transformation of the lattice $\boldsymbol{x}$ we need three new ingredients: $\phi, \boldsymbol{x}_{\mathrm{C}}$ and ϕ_{C}. In looking for the Ribaucour reduction of the fundamental transformation we can use the additional information:
(i) the initial lattice $\boldsymbol{x}$ satisfies the quadratic constraint (3),
(ii) the final lattice $\mathcal{R}(\boldsymbol{x})$ should satisfy the same constraint as well.

This should allow to reduce the number of the necessary data and, indeed, to find the Ribaucour transformation it is enough to know the Combescure transformation vector $\boldsymbol{x}_{\mathrm{C}}$ only.

Proposition 2. *The Ribaucour reduction $\mathcal{R}(\boldsymbol{x})$ of the fundamental transformation of the quadrilateral lattice $\boldsymbol{x}$ subjected to quadratic constraint* (3) *is determined by the Combescure transformation vector $\boldsymbol{x}_{\mathrm{C}}$, provided that $\boldsymbol{x}_{\mathrm{C}}$ is not anihilated by the bilinear form Q of the constraint*

$$\boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}} Q \boldsymbol{x}_{\mathrm{C}} \neq 0. \tag{8}$$

The functions ϕ and ϕ_{C} entering in formula (4) *are then given by*

$$\phi = 2\boldsymbol{x}^{\mathrm{t}} Q \boldsymbol{x}_{\mathrm{C}} + \boldsymbol{a}^{\mathrm{t}} \boldsymbol{x}_{\mathrm{C}}, \tag{9}$$

$$\phi_{\mathrm{C}} = \boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}} Q \boldsymbol{x}_{\mathrm{C}}. \tag{10}$$

Proof. Application of the partial difference operator Δ_i to the quadratic constraint (3) gives

$$(T_i \boldsymbol{x}^{\mathrm{t}}) Q (\Delta_i \boldsymbol{x}) + \boldsymbol{x}^{\mathrm{t}} Q (\Delta_i \boldsymbol{x}) + \boldsymbol{a}^{\mathrm{t}} (\Delta_i \boldsymbol{x}) = 0. \tag{11}$$

Applying Δ_j, $j \neq i$ to Eq. (11) and making use of Eqs. (1) and (11), we obtain

$$(T_i \Delta_j \boldsymbol{x}^{\mathrm{t}}) Q (T_j \Delta_i \boldsymbol{x}) + (\Delta_j \boldsymbol{x}^{\mathrm{t}}) Q (\Delta_i \boldsymbol{x}) (1 + T_i A_{ij} + T_j A_{ji}) = 0. \tag{12}$$

We recall (see [18] for details) that, given Combescure transformation vector $\boldsymbol{x}_{\mathrm{C}}$, it satisfies the Laplace equation:

$$\Delta_i \Delta_j \boldsymbol{x}_{\mathrm{C}} = (T_i A_{ij}^{\mathrm{C}}) \Delta_i \boldsymbol{x}_{\mathrm{C}} + (T_j A_{ji}^{\mathrm{C}}) \Delta_j \boldsymbol{x}_{\mathrm{C}}, \quad i \neq j, \quad i, j = 1, \ldots, N, \tag{13}$$

with

$$A_{ij}^{\mathrm{C}} = \frac{T_j \sigma_j}{\sigma_i} A_{ij}. \tag{14}$$

We will show that the function ϕ_{C}, defined in (10), satisfies the Laplace equation (13) of the lattice $\boldsymbol{x}_{\mathrm{C}}$. Indeed, the difference operator Δ_i acting on ϕ_{C} gives

$$\Delta_i \phi_{\mathrm{C}} = (T_i \boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}}) Q(\Delta_i \boldsymbol{x}_{\mathrm{C}}) + \boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}} Q(\Delta_i \boldsymbol{x}_{\mathrm{C}}).$$

Applying Δ_j, $j \neq i$, to the above equation and making use of Eq. (13) we obtain

$$\begin{aligned}&\Delta_i \Delta_j \phi_{\mathrm{C}} - (T_i A_{ij}^{\mathrm{C}}) \Delta_i \phi_{\mathrm{C}} - (T_j A_{ji}^{\mathrm{C}}) \Delta_j \phi_{\mathrm{C}} \\ &\quad = (T_i \Delta_j \boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}}) Q(T_j \Delta_i \boldsymbol{x}_{\mathrm{C}}) + (\Delta_j \boldsymbol{x}_{\mathrm{C}}^{\mathrm{t}}) Q(\Delta_i \boldsymbol{x}_{\mathrm{C}})(1 + T_i A_{ij}^{\mathrm{C}} + T_j A_{ji}^{\mathrm{C}}).\end{aligned} \tag{15}$$

Making use of Eqs. (5) and (12) we transform Eq. (15) to the form

$$\begin{aligned}&\Delta_i \Delta_j \phi_{\mathrm{C}} - (T_i A_{ij}^{\mathrm{C}}) \Delta_i \phi_{\mathrm{C}} - (T_j A_{ji}^{\mathrm{C}}) \Delta_j \phi_{\mathrm{C}} \\ &\quad = (\Delta_j \boldsymbol{x}^{t}) Q(\Delta_i \boldsymbol{x})[(T_i \sigma_i)(T_j \sigma_j)(1 + T_i A_{ij}^{\mathrm{C}} + T_j A_{ji}^{\mathrm{C}}) \\ &\qquad - (T_i T_j \sigma_i)(T_i T_j \sigma_j)(1 + T_i A_{ij} + T_j A_{ji})];\end{aligned}$$

the expression in square brackets vanishes due to (14) and (6), which shows that the function ϕ_{C} does satisfy the Laplace Eq. (13).

It is easy to see that in order to satisfy constraint (3) the function ϕ must be defined as in (9). Moreover, by direct verification one can check that ϕ and ϕ_{C} are connected by Eq. (7), which also implies that ϕ satisfies the Laplace Eq. (1) of the lattice $\boldsymbol{x}$. □

Remark. *Condition* (8) *is satisfied, in particular, when the quadric has non-degenerate and definite bilinear form.*

Let us discuss the geometric meaning of the algebraic results obtained above. The congruence $\mathfrak{l}$ of the fundamental transformation is defined once the Combescure transformation vector is given; moreover, any generic congruence conjugate to $\boldsymbol{x}$ can be obtained in this way (for details, see [18]). The points of the transformed lattice $\mathcal{R}(\boldsymbol{x})$ belong to the lines of the congruence and to the quadric $\mathcal{Q}$. Therefore, we can formulate the following analog of the Ribaucour theorem [21], which also follows directly from Lemma 1.

Proposition 3. *If a congruence is conjugate to a quadrilateral lattice contained in a quadric, and if each line of the congruence meets the quadric just in two distinct points, then the second intersection of the congruence and the quadric is also a quadrilateral lattice conjugate to the congruence.*

Remark. *If a line and a quadric hypersurface have non-trivial intersection, then they have exactly two points in common (counting with multiplicities and points at infinity) or, alternatively, the line is contained in the quadric.*

3.2. Superposition of Ribaucour transformations

In this section we consider vectorial Ribaucour transformations, which are nothing else but superpositions of the Ribaucour transformations with appropriate transformation data.

We first recall [18,29] the necessary material concerning the vectorial fundamental transformations. Consider $K \geq 1$ fundamental transformations $\mathcal{F}_k(\boldsymbol{x}), k = 1, \ldots, K$, of the quadrilateral lattice $\boldsymbol{x} \subset \mathbb{R}^M$, which are built from K solutions $\phi^k, k = 1, \ldots, K$ of the Laplace equation of the lattice $\boldsymbol{x}$ and K Combescure transformation vectors $\boldsymbol{x}_{\mathrm{C},k}$, where

$$\Delta_i \boldsymbol{x}_{\mathrm{C},k} = (T_i \sigma_{i,k}) \Delta_i \boldsymbol{x}, \quad i = 1, \ldots, N, \quad k = 1, \ldots, K,$$

and $\sigma_{i,k}$ satisfy equations

$$\Delta_j \sigma_{i,k} = A_{ij}(T_j \sigma_{j,k} - T_j \sigma_{i,k}), \quad i \neq j;$$

finally, we are given also K functions $\phi^k_{\mathrm{C},k}$, which satisfy

$$\Delta_i \phi^k_{\mathrm{C},k} = (T_i \sigma_{i,k}) \Delta_i \phi^k.$$

We arrange functions ϕ^k in the K component vector $\phi = (\phi^1, \ldots, \phi^K)^{\mathrm{t}}$, similarily, we arrange the Combescure transformation vectors $\boldsymbol{x}_{\mathrm{C},k}$ into $M \times K$ matrix $\boldsymbol{X}_{\mathrm{C}} = (\boldsymbol{x}_{\mathrm{C},1}, \ldots, \boldsymbol{x}_{\mathrm{C},K})$; moreover, we introduce the $K \times K$ matrix $\boldsymbol{\Phi}_{\mathrm{C}} = (\phi_{\mathrm{C},1}, \ldots, \phi_{\mathrm{C},K})$, whose columns are the K component vectors $\phi_{\mathrm{C},k} = (\phi^1_{\mathrm{C},k}, \ldots, \phi^K_{\mathrm{C},k})^{\mathrm{t}}$ being the Combescure transforms of ϕ

$$\Delta_i \phi_{\mathrm{C},k} = (T_i \sigma_{i,k}) \Delta_i \phi. \tag{16}$$

Remark. *The diagonal part of $\boldsymbol{\Phi}_{\mathrm{C}}$ is fixed by the initial fundamental transformations. To find the off-diagonal part of $\boldsymbol{\Phi}_{\mathrm{C}}$ we integrate Eq.* (16) *introducing $K(K-1)$ arbitrary constants.*

One can show that the vectorial fundamental transformation $\mathcal{F}(\boldsymbol{x})$ of the quadrilateral lattice $\boldsymbol{x}$, which is defined as

$$\mathcal{F}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{X}_{\mathrm{C}} \boldsymbol{\Phi}_{\mathrm{C}}^{-1} \phi, \tag{17}$$

is again quadrilateral lattice. Moreover, the vectorial transformation is superposition of the fundamental transformations

$$\mathcal{F}(\boldsymbol{x}) = (\mathcal{F}_{k_1} \circ \mathcal{F}_{k_2} \circ \cdots \circ \mathcal{F}_{k_K})(\boldsymbol{x}), \quad k_i \neq k_j \ \text{ for } \ i \neq j,$$

and does not depend on the order in which the transformations are taken. In applying the fundamental transformations at the intermediate stages the transformation data should be suitably transformed as well. To prove the superposition and permutability statements it is important to notice that the following basic fact holds:

Lemma 2. *Assume the following splitting of the data of the vectorial fundamental transformation*

$$\phi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}, \qquad \boldsymbol{X}_{\mathrm{C}} = (\boldsymbol{X}_{\mathrm{C}(1)}, \boldsymbol{X}_{\mathrm{C}(2)}), \qquad \boldsymbol{\Phi}_{\mathrm{C}} = \begin{pmatrix} \boldsymbol{\Phi}^{(1)}_{\mathrm{C}(1)} & \boldsymbol{\Phi}^{(1)}_{\mathrm{C}(2)} \\ \boldsymbol{\Phi}^{(2)}_{\mathrm{C}(1)} & \boldsymbol{\Phi}^{(2)}_{\mathrm{C}(2)} \end{pmatrix},$$

associated with partition $K = K_1 + K_2$. *Then the vectorial fundamental transformation* $\mathcal{F}(\boldsymbol{x})$ *is equivalent to the following superposition of vectorial fundamental transformations:*

1. *Transformation* $\mathcal{F}_{(1)}(\boldsymbol{x})$ *with the data* $\phi^{(1)}, X_{\mathrm{C}(1)}, \Phi^{(1)}_{\mathrm{C}(1)}$*:*

$$\mathcal{F}_{(1)}(\boldsymbol{x}) = \boldsymbol{x} - X_{\mathrm{C}(1)}(\Phi^{(1)}_{\mathrm{C}(1)})^{-1}\phi^{(1)}.$$

2. *Application on the result obtained in point* 1, *transformation* $\mathcal{F}_{(2)}$ *with the data transformed by the transformation* $\mathcal{F}_{(1)}$ *as well*

$$\mathcal{F}_{(2)}(\mathcal{F}_{(1)}(\boldsymbol{x})) = \mathcal{F}_{(1)}(\boldsymbol{x}) - \mathcal{F}_{(1)}(X_{\mathrm{C}(2)})(\mathcal{F}_{(1)}(\Phi^{(2)}_{\mathrm{C}(2)}))^{-1}\mathcal{F}_{(1)}(\phi^{(2)}),$$

where:

$$\mathcal{F}_{(1)}(X_{\mathrm{C}(2)}) = X_{\mathrm{C}(2)} - X_{\mathrm{C}(1)}(\Phi^{(1)}_{\mathrm{C}(1)})^{-1}\Phi^{(1)}_{\mathrm{C}(2)}, \tag{18}$$

$$\mathcal{F}_{(1)}(\phi^{(2)}) = \phi^{(2)}\Phi^{(2)}_{\mathrm{C}(1)}(\Phi^{(1)}_{\mathrm{C}(1)})^{-1}\phi^{(1)}, \tag{19}$$

$$\mathcal{F}_{(1)}(\Phi^{(2)}_{\mathrm{C}(2)}) = \Phi^{(2)}_{\mathrm{C}(2)} - \Phi^{(2)}_{\mathrm{C}(1)}(\Phi^{(1)}_{\mathrm{C}(1)})^{-1}\Phi^{(1)}_{\mathrm{C}(2)}. \tag{20}$$

Corollary 3. *For any* $L = 0, \ldots, K-2$, *the points* $\boldsymbol{x}' = (\mathcal{F}_{k_1} \circ \cdots \circ \mathcal{F}_{k_L})(\boldsymbol{x})$, $\mathcal{F}_{k_{L+1}}(\boldsymbol{x}')$, $\mathcal{F}_{k_{L+2}}(\boldsymbol{x}')$, $(\mathcal{F}_{k_{L+1}} \circ \mathcal{F}_{k_{L+2}})(\boldsymbol{x}')$ *are coplanar.*

Remark. *The* $K(K-1)$ *constants of integration in the off-diagonal part of* Φ_{C} *are used to construct "initial quadrilaterals", i.e., the integration constants in* $\phi^{\ell}_{\mathrm{C},k}$ *and* $\phi^{k}_{\mathrm{C},\ell}$ $(k \neq \ell)$ *fix the position of* $(\mathcal{F}_k \circ \mathcal{F}_\ell)(\boldsymbol{x})$ *on the plane passing through* $\boldsymbol{x}$, $\mathcal{F}_k(\boldsymbol{x})$ *and* $\mathcal{F}_\ell(\boldsymbol{x})$. *The rest of the construction is by linear algebra and is the direct consequence of the geometric integrability scheme (Theorem* 1*). Any point* $\boldsymbol{x}$ *of the initial lattice, together with its images under all possible superpositions* $\mathcal{F}_{k_1}(\boldsymbol{x}), \ldots, (\mathcal{F}_{k_1} \circ \mathcal{F}_{k_2})(\boldsymbol{x}), \ldots, (\mathcal{F}_{k_1} \circ \cdots \circ \mathcal{F}_{k_K})(\boldsymbol{x})$, *form a network of the type of* K*-hypercube. Different paths from* $\boldsymbol{x}$ *to the opposite diagonal vertex* $\mathcal{F}(\boldsymbol{x})$ *represent various ordering of the fundamental transformations in the final superposition.*

To find the Ribaucour reduction of the vectorial fundamental transformation we can use results of Section 3.1 to obtain.

$$\phi^k = 2\boldsymbol{x}^{\mathrm{t}} Q \boldsymbol{x}_{\mathrm{C},k} + \boldsymbol{a}^{\mathrm{t}}\boldsymbol{x}_{\mathrm{C},k}, \tag{21}$$

$$\phi^k_{\mathrm{C},k} = \boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},k} Q \boldsymbol{x}_{\mathrm{C},k}. \tag{22}$$

Eqs. (16) and (21) lead to

$$\Delta_i(\phi^k_{\mathrm{C},\ell} + \phi^\ell_{\mathrm{C},k}) = (T_i\boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},k} + \boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},k}) Q(\Delta_i \boldsymbol{x}_\ell) + (T_i\boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},\ell} + \boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},\ell}) Q(\Delta_i \boldsymbol{x}_k),$$

which implies that

$$\phi^k_{\mathrm{C},\ell} + \phi^\ell_{\mathrm{C},k} = 2\boldsymbol{x}^{\mathrm{t}}_{\mathrm{C},k} Q \boldsymbol{x}_{\mathrm{C},\ell}; \tag{23}$$

the constant of integration was found from condition $(\mathcal{R}_k \circ \mathcal{R}_\ell)(\boldsymbol{x}) \subset \mathcal{Q}$.

Proposition 4. *The vectorial Ribaucour transformation $\boldsymbol{\mathcal{R}}$, i.e., the reduction of the vectorial fundamental transformation* (17) *compatible with quadratic constraint* (3), *is given by the following constraints:*

$$\phi^{\mathrm{t}} = 2\boldsymbol{x}^{\mathrm{t}} Q X_{\mathrm{C}} + \boldsymbol{a}^{\mathrm{t}} X_{\mathrm{C}}, \tag{24}$$

$$\boldsymbol{\Phi}_{\mathrm{C}} + \boldsymbol{\Phi}_{\mathrm{C}}^{\mathrm{t}} = 2X_{\mathrm{C}}^{\mathrm{t}} Q X_{\mathrm{C}}. \tag{25}$$

Proof. Eqs. (24) and (25) are just compact forms of Eqs. (21)–(23), which assert that, if $\boldsymbol{x} \subset \mathcal{Q}$ then $\mathcal{R}_k(\boldsymbol{x}) \subset \mathcal{Q}$, $(\mathcal{R}_k \circ \mathcal{R}_\ell)(\boldsymbol{x}) \subset \mathcal{Q}$ as well. Moreover, since Corollary 3 still holds, then from Lemma 1 it follows that, at each step of the superposition, the lattice $(\mathcal{R}_{k_1} \circ \cdots \circ \mathcal{R}_{k_L})(\boldsymbol{x})$ is also contained in the quadric, which implies the stated result.

The algebraic verification that $\boldsymbol{\mathcal{R}}(\boldsymbol{x})$ belongs to the quadric $\mathcal{Q}$ is also immediate. Using condition (3) we obtain

$$\begin{aligned}
&\boldsymbol{\mathcal{R}}(\boldsymbol{x})^{\mathrm{t}} Q \boldsymbol{\mathcal{R}}(\boldsymbol{x}) + \boldsymbol{a}^{\mathrm{t}} \boldsymbol{\mathcal{R}}(\boldsymbol{x}) + c \\
&\quad = \phi^{\mathrm{t}} (\boldsymbol{\Phi}_{\mathrm{C}}^{\mathrm{t}})^{-1} X_{\mathrm{C}}^{\mathrm{t}} Q X_{\mathrm{C}} \boldsymbol{\Phi}_{\mathrm{C}}^{-1} \phi - \boldsymbol{x}^{\mathrm{t}} Q X_{\mathrm{C}} \boldsymbol{\Phi}_{\mathrm{C}}^{-1} \phi \\
&\qquad - \phi^{\mathrm{t}} (\boldsymbol{\Phi}_{\mathrm{C}}^{\mathrm{t}})^{-1} X_{\mathrm{C}}^{\mathrm{t}} Q \boldsymbol{x} - \boldsymbol{a}^{\mathrm{t}} X_{\mathrm{C}} \boldsymbol{\Phi}_{\mathrm{C}}^{-1} \phi,
\end{aligned}$$

which vanishes due to Eqs. (24) and (25) and the following identity:

$$\boldsymbol{a}^{\mathrm{t}} X_{\mathrm{C}} \boldsymbol{\Phi}_{\mathrm{C}}^{-1} \phi = \phi^{\mathrm{t}} (\boldsymbol{\Phi}_{\mathrm{C}}^{\mathrm{t}})^{-1} X_{\mathrm{C}}^{\mathrm{t}} \boldsymbol{a}. \qquad \square$$

Notice that proving geometrically the above proposition we proved also the analog of Lemma 2 (which we would like to prove algebraically as well).

Proposition 5. *Assume the following splitting of the data of the vectorial Ribaucour transformation:*

$$\phi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}, \qquad X_{\mathrm{C}} = (X_{\mathrm{C}(1)}, X_{\mathrm{C}(2)}), \qquad \boldsymbol{\Phi}_{\mathrm{C}} = \begin{pmatrix} \boldsymbol{\Phi}_{\mathrm{C}(1)}^{(1)} & \boldsymbol{\Phi}_{\mathrm{C}(2)}^{(1)} \\ \boldsymbol{\Phi}_{\mathrm{C}(1)}^{(2)} & \boldsymbol{\Phi}_{\mathrm{C}(2)}^{(2)} \end{pmatrix}, \tag{26}$$

associated with partition $K = K_1 + K_2$. Then the vectorial Ribaucour transformation $\boldsymbol{\mathcal{R}}(\boldsymbol{x})$ is equivalent to the following superposition of vectorial Ribaucour transformations:

1. *Transformation $\boldsymbol{\mathcal{R}}_{(1)}(\boldsymbol{x})$ with the data $\phi^{(1)}, X_{\mathrm{C}(1)}, \boldsymbol{\Phi}_{\mathrm{C}(1)}^{(1)}$.*
2. *Application on the result obtained in point* 1, *the transformation $\boldsymbol{\mathcal{R}}_{(2)}$ with the data $\boldsymbol{\mathcal{R}}_{(1)}(X_{\mathrm{C}(2)}), \boldsymbol{\mathcal{R}}_{(1)}(\boldsymbol{\Phi}_{\mathrm{C}(2)}^{(2)}), \boldsymbol{\mathcal{R}}_{(1)}(\phi^{(2)})$ given by $\mathcal{R}$- analogs of formulas* (18)–(20).

Proof. We have to show that the data of both transformations satisfy constraints (24) and (25). Since the data (26) do satisfy the constraints we have:

$$\phi_{(i)}^{\mathrm{t}} = 2\boldsymbol{x}^{\mathrm{t}} Q X_{\mathrm{C}(i)} + \boldsymbol{a}^{\mathrm{t}} X_{\mathrm{C}(i)}, \quad i = 1, 2, \tag{27}$$

$$\boldsymbol{\Phi}_{\mathrm{C}(i)}^{(i)} + (\boldsymbol{\Phi}_{\mathrm{C}(i)}^{(i)})^{\mathrm{t}} = 2X_{\mathrm{C}(i)}^{\mathrm{t}} Q X_{\mathrm{C}(i)}, \tag{28}$$

$$\boldsymbol{\Phi}_{\mathrm{C}(2)}^{(1)} + (\boldsymbol{\Phi}_{\mathrm{C}(1)}^{(2)})^{\mathrm{t}} = 2X_{\mathrm{C}(1)}^{\mathrm{t}} Q X_{\mathrm{C}(2)}, \tag{29}$$

this leads immediately to conclusion that the transformation 1 is the Ribaucour transformation. Verification that the data of the transformation of point 2 satisfy constraints (24) and (25) can be done by straightforward algebra. □

Corollary 4. *Obviously, one can reverse the order of the two transformations (keeping in mind suitable transformation of their data). Moreover, the above result implies that assuming a general splitting $K = K_1 + \cdots + K_P$ the final result does not depend on the order in which the transformations are made.*

We finally remark that recursive application of the fundamental transformations can be considered [18] as generating new dimensions of the quadrilateral lattice $\boldsymbol{x}$. In this context, the Ribaucour transformations generate new dimensions of the lattice subjected to the quadratic constraint. This interpretation remains valid also in the limit from the quadrilateral lattice $\boldsymbol{x} \subset \mathcal{Q}$ to the multiconjugate net $\boldsymbol{x} \subset \mathcal{Q}$. Therefore, the Ribaucour transformations of multiconjugate nets subjected to quadratic constraints generate their natural, geometricaly distinguished, integrable discrete analogs.

4. Circular lattices and their Ribaucour transformation

In this section we illustrate the quadratic reduction approach on a simple example when the quadric $\mathcal{Q}$ is the M-dimensional sphere $\mathbb{S}^M \subset \mathbb{E}^{M+1}$ of radius 1; the bilinear form Q is just the standard scalar product "$\cdot$" in the $(M+1)$-dimensional Euclidean space (we add one dimension for convenience), and the quadratic constraint (3) takes the form $\boldsymbol{x} \cdot \boldsymbol{x} = 1$.

4.1. Circular lattices and Möbius geometry

Given the point $N \in \mathbb{S}^M$ (called the North Pole), consider the hyperplane $\mathbb{T} \simeq \mathbb{E}^M$ bisecting the sphere and orthogonal to N. In standard way we define the stereographic projection St: $\mathbb{S}^M \to \mathbb{T} \cup \{\infty\}$ such that for all $\boldsymbol{x} = (x^0, \vec{\boldsymbol{x}}) \in \mathbb{S}^M \setminus \{N\}, \boldsymbol{y} = \mathrm{St}(\boldsymbol{x})$ is the unique intersection point of the line $\langle N, \boldsymbol{x}\rangle$ with the hyperplane $\mathbb{T}$:

$$
\begin{aligned}
\boldsymbol{y} &= \mathrm{St}(\boldsymbol{x}) = \frac{\vec{\boldsymbol{x}}}{1 - x^0},\\
\boldsymbol{x} &= (x^0, \vec{\boldsymbol{x}}) = \mathrm{St}^{-1}(\boldsymbol{y}) = \left(\frac{|\boldsymbol{y}|^2 - 1}{|\boldsymbol{y}|^2 + 1}, \frac{2\boldsymbol{y}}{|\boldsymbol{y}|^2 + 1} \right), \quad |\boldsymbol{y}|^2 = \boldsymbol{y} \cdot \boldsymbol{y},
\end{aligned} \tag{30}
$$

and the North Pole is mapped into the infinity point ∞.

We recall the basic property of the stereographic projection [31] which is an important tool in the conformal (or Möbius) geometry.

Lemma 3. *Circles of the sphere $\mathbb{S}^M$ are mapped in the stereographic projection into circles or straight lines (i.e., circles passing through the infinity point) of the hyperplane $\mathbb{T} \simeq \mathbb{E}^M$.*

Since the intersection of the plane of any elementary quadrilateral of $\boldsymbol{x}$ with the sphere $\mathbb{S}^M$ is a circle we have therefore:

Proposition 6. *Quadrilateral lattices in the sphere $\mathbb{S}^M$ are mapped in the stereographic projection into multidimensional circular lattices in $\mathbb{E}^M$; conversely, any circular lattice $\mathbb{E}^M$ can be obtained in this way.*

Remark. *The Möbius geometry studies invariants of the transformations of Euclidean space, which map circles into circles. The Möbius transformations act, therefore, within the space of circular lattices, like the projective transformations act within the space of quadrilateral lattices (see* [13, 16]). *One can identify two circular lattices which are connected by a Möbius transformation and study the circular lattices in the Möbius geometry approach.*

Proposition 6 provides a convenient characterization of the circularity constraint [25].

Theorem 3. *The quadrilateral lattice $\boldsymbol{y} \subset \mathbb{E}^M$ is circular if and only if the scalar function $r = |\boldsymbol{y}|^2$ is a solution of the Laplace equation of the lattice $\boldsymbol{y}$.*

Proof. The quadrilateral lattice $\boldsymbol{y}$, satisfying the following system of Laplace equations:

$$\Delta_i \Delta_j \boldsymbol{y} = (T_i B_{ij}) \Delta_i \boldsymbol{y} + (T_j B_{ji}) \Delta_j \boldsymbol{y}, \quad i \neq j, \quad i, j = 1, \ldots, N, \tag{31}$$

is circular if and only if the lattice $\boldsymbol{x} = \mathrm{St}^{-1}(\boldsymbol{y}) \subset \mathbb{S}^M \subset \mathbb{E}^{M+1}$ is quadrilateral, i.e., $\boldsymbol{x}$ satisfies the Laplace Eq. (1). Obviously, if $\boldsymbol{x}$ is quadrilateral, then the $\mathbb{E}^M$ part of $\boldsymbol{x}$, i.e., $\vec{\boldsymbol{x}} = 2\boldsymbol{y}/(|\boldsymbol{y}|^2 + 1)$, satisfies Eq. (1) as well.

The idea of the proof is based on the following observation. We recall (see [16]) that, if $\boldsymbol{y}$ satisfies Eqs. (31), then, for any gauge function ρ, the new lattice $\tilde{\boldsymbol{y}} = \rho^{-1}\boldsymbol{y}$ satisfies equations

$$\Delta_i \Delta_j \tilde{\boldsymbol{y}} = (T_i \tilde{B}_{ij}) \Delta_i \tilde{\boldsymbol{y}} + (T_j \tilde{B}_{ji}) \Delta_j \tilde{\boldsymbol{y}} + \tilde{C}_{ij} \tilde{\boldsymbol{y}}, i \neq j, \quad i, j = 1, \ldots, N,$$

with

$$\tilde{B}_{ij} = (T_j \rho)^{-1} (B_{ij} - \Delta_{j\rho}), \quad i \neq j, \quad i, j = 1, \ldots, N,$$
$$\tilde{C}_{ij} = (T_i T_j \rho)^{-1} (-\Delta_i \Delta_{j\rho} + (T_i B_{ij}) \Delta_i \rho + (T_j B_{ij}) \Delta_j \rho).$$

The rest of the proof follows from the fact that, in our case, $\rho = (|\boldsymbol{y}|^2 + 1)/2$ and $\tilde{C}_{ij} = 0$. □

Remark. *In the continuous context, the direct analog of Theorem* 3 *leads immediately to orthogonality of the intersecting conjugate coordinate lines* [21]. *The above characterization of circular lattices was postulated in* [25] *where its relation to geometry was made via another (equivalent) form of the circularity constraint* [17].

4.2. Ribaucour transformation of the circular lattices

We recall that the fundamental transformation $\mathcal{F}(y)$ of the quadrilateral lattice y generates quadrilateral strip with N-dimensional basis y and transversal direction $\mathcal{F}$ (the quadrilaterals $\{y, T_i y, \mathcal{F}(y), T_i\mathcal{F}(y)\}$, $i = 1, \dots, N$, are planar as well). When y is subjected to the circularity condition, then it is natural to consider only such fundamental transformations which act within the space of circular lattices [25].

Definition. The Ribaucour transformation $\mathcal{R}^\circ(y)$ of the circular lattice y is a fundamental transformation such that all the strip with N-dimensional basis y and transversal direction $\mathcal{R}^\circ$ is made out of circular quadrilaterals.

Remark. *It is not enough to define the Ribaucour transformation $\mathcal{R}^\circ(y)$ of the circular lattice y as a fundamental transformation such that the transformed lattice is circular as well.*

In this section we present the Ribaucour transformation of multidimensional circular lattices from the point of view of quadratic reductions. Given circular lattice $y \subset \mathbb{E}^M$, we apply to $x = \mathrm{St}^{-1}(y) \subset \mathbb{S}^M$ the Ribaucour transformation $\mathcal{R}$, defined in Section 3, obtaining the new lattice $\mathcal{R}(x) \subset \mathbb{S}^M$. Since, for points in the sphere, planarity implies circularity we conclude that the quadrilaterals $\mathrm{St}(\{x, T_i x, \mathcal{R}(x), T_i\mathcal{R}(x)\})$ are circular. This observation, together with Lemma 3 and Proposition 6, leads to the following result.

Proposition 7. *The transformation $\mathrm{St}(\mathcal{R}(\mathrm{St}^{-1}(y)))$ is a Ribaucour transformation of the circular lattice y; conversely, any Ribaucour transformation $\mathcal{R}^\circ(y)$ of the circular lattice y can be obtained in this way.*

Corollary 5. *One can extend, via formula* (30), *the stereographic projection* St *to the projection* P *of $\mathbb{E}^{M+1}$ on $\mathbb{T}$ with the center in N. In this way the lines $\mathfrak{l}$ of the congruence of the transformation $\mathcal{R}$ are mapped into the lines $\mathfrak{l}^0 = \mathrm{P}(\mathfrak{l})$ of the congruence of the transformation $\mathcal{R}^\circ$. However, since the central projection does not preserve parallelism, it cannot be used directly to define the Combescure transformation vector y_C, from given x_C; one needs some rescaling.*

The rest of this section is devoted to "algebraization" of the above geometric observations.

Consider the circular lattice $y \subset \mathbb{E}^M$ and its image in the Möbius sphere $x = \mathrm{St}^{-1}(y) \subset \mathbb{S}^M$. The Ribaucour transformation $\mathcal{R}(x)$ of x

$$\mathcal{R}(x) = x - \frac{2x \cdot x_C}{x_C \cdot x_C} x_C$$

is mapped in the steroegraphic projection to

$$\mathrm{St}(\mathcal{R}(x)) = y - (2y \cdot \vec{x}_C + x_C^0(|y|^2 - 1)) \frac{x_C^0 y + \vec{x}_C}{|x_C^0 y + \vec{x}_C|^2}.$$

One can directly verify that the function

$$y_C = x_C^0 y + \vec{x}_C,$$

is the Combescure transformation vector of the circular lattice y

$$\Delta_i y_C = (T_i \varrho_i) \Delta_i y,$$

with

$$\varrho_i = x_C^0 + \frac{2\sigma_i}{|y|^2 + 1},$$

and $|y_C|^2$ satisfies the Laplace equation of the lattice y_C. Moreover, the function

$$\psi = 2y \cdot \vec{x}_C + x_C^0(|y|^2 - 1) = 2y \cdot y_C - x_C^0(|y|^2 + 1)$$

satisfies equation

$$\Delta_i \psi = \frac{1}{T_i \varrho_i} \Delta_i (|y_C|^2) = \Delta_i y \cdot (T_i y_C + y_C). \tag{32}$$

Putting these facts together we arrive to the following characterization of the Ribaucour transformation of circular lattices [25].

Theorem 4. *The Ribaucour transformation of the circular lattice $y \subset \mathbb{E}^M$ reads*

$$\mathcal{R}^\circ(y) = y - \frac{\psi}{\psi_C} y_C, \tag{33}$$

where y_C is the Combescure vector of y, $\psi_C = |y_C|^2$, and ψ is a solution of Eq. (32).

We would like to add a few remarks, which follow directly from the above reasoning, or can be easily verified.

Corollary 6.
(i) *When P is the projection defined in Corollary* 5 *then*

$$P(x + x_C) - P(x) = \frac{y_C}{(1 - x^0 - x_C^0)}.$$

(ii) *Then function ψ can be written in the form*

$$\psi = 2y \cdot y_C - (|y|^2)_C. \tag{34}$$

Corollary 7.
(i) *In the simplest case, when $y_C = y$, then $\psi = |y|^2 - a$, where $a =$* const, *and the corresponding Ribaucour transformation is the inversion*

$$\mathcal{R}^\circ(y) = \mathcal{I}_a(y) = a \frac{y}{|y|^2}.$$

(ii) *The Combescure transformation of a circular lattice is circular lattice as well.*
(iii) *The Ribaucour transformation can be decomposed into superposition of two Combescure transformations and inversion:*

$$\begin{array}{ccc} \boldsymbol{y} & \stackrel{\mathcal{R}^\circ}{\longleftrightarrow} & \mathcal{R}^\circ(\boldsymbol{y}) \\ \mathcal{C}\updownarrow & & \updownarrow\mathcal{C} \\ \boldsymbol{y}_{\mathrm{C}} & \stackrel{\mathcal{I}}{\longleftrightarrow} & \mathcal{I}(\boldsymbol{y}_{\mathrm{C}}) = (\mathcal{R}^\circ(\boldsymbol{y}))_{\mathrm{C}}. \end{array}$$

Remark. *We recall that, in the case of the fundamental transformation of the quadrilateral lattice $\boldsymbol{x}$, the Combescure transformation vectors $\boldsymbol{x}_{\mathrm{C}}$ and $(\mathcal{F}(\boldsymbol{x}))_{\mathrm{C}}$ (they define the same congruence but from the point of view of two different lattices) are related by the radial transformation* [18].

For completness, we present also the vectorial Ribaucour transformation of circular lattices. Consider K Ribaucour transformations of the circular lattice $\boldsymbol{y}$, which are defined by the Combescure vectors $\boldsymbol{y}_{\mathrm{C},k}$ and the corresponding transforms $r_{\mathrm{C},k}$ of $r = |\boldsymbol{y}|^2$:

$$\Delta_i \begin{pmatrix} \boldsymbol{y}_{\mathrm{C},k} \\ r_{\mathrm{C},k} \end{pmatrix} = (T_i \varrho_{i,k}) \Delta_i \begin{pmatrix} \boldsymbol{y} \\ r \end{pmatrix},$$

which we arrange in $M \times K$ matrix $\boldsymbol{Y}_{\mathrm{C}} = (\boldsymbol{y}_{\mathrm{C},1}, \ldots, \boldsymbol{y}_{\mathrm{C},K})$ and the row vector $\boldsymbol{r}_{\mathrm{C}} = (r_{\mathrm{C},1}, \ldots, r_{\mathrm{C},K})$. The corresponding vector ψ (see Eq. (34)) has components

$$\psi^k = 2\boldsymbol{y} \cdot \boldsymbol{y}_{\mathrm{C},k} - r_{\mathrm{C},k}.$$

Using Eqs. (33) and (32) and the condition that $(\mathcal{R}^\circ_k \circ \mathcal{R}^\circ_\ell)(\boldsymbol{y})$ belongs to the circle passing through the points $\boldsymbol{y}$, $\mathcal{R}^\circ_k(\boldsymbol{y})$ and $\mathcal{R}^\circ_\ell(\boldsymbol{y})$, one can show that the components of the matrix $\boldsymbol{\Psi}_{\mathrm{C}}$ being defined as

$$\Delta_i \psi^k_{\mathrm{C},\ell} = (T_i \rho_{i,\ell}) \Delta_i \psi^k,$$

satisfy condition

$$\psi^k_{\mathrm{C},\ell} + \psi^\ell_{\mathrm{C},k} = 2\boldsymbol{y}_{\mathrm{C},k} \cdot \boldsymbol{y}_{\mathrm{C},\ell}.$$

Finally, we present the "circular" analogs of Propositions 4 and 5 of Section 3.2.

Proposition 8. *The vectorial Ribaucour transformation $\boldsymbol{\mathcal{R}}^\circ$, i.e., the reduction of the vectorial fundamental transformation* (17) *compatible with the circularity constraint, is given by*

$$\boldsymbol{\mathcal{R}}^\circ(\boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{Y}_{\mathrm{C}} \boldsymbol{\Psi}_{\mathrm{C}}^{-1} \boldsymbol{\psi},$$

with the following constraints

$$\boldsymbol{\psi}^{\mathrm{t}} = 2\boldsymbol{y} \cdot \boldsymbol{Y}_{\mathrm{C}} - \boldsymbol{r}_{\mathrm{C}},$$
$$\boldsymbol{\Psi}_{\mathrm{C}} + \boldsymbol{\Psi}_{\mathrm{C}}^{\mathrm{t}} = 2\boldsymbol{Y}_{\mathrm{C}} \cdot \boldsymbol{Y}_{\mathrm{C}}.$$

Proof. We have to show that the lattice $\mathcal{R}^\circ(y)$ is circular, i.e., the function $|\mathcal{R}^\circ(y)|^2$ is a solution of the Laplace equation of the lattice $\mathcal{R}^\circ(y)$.

First notice that, since r satisfies the Laplace equation of the lattice y, the function

$$\mathcal{R}^\circ(r) = r - r_C \boldsymbol{\Psi}_C^{-1} \psi$$

is a solution of the Laplace equation of the lattice $\mathcal{R}^\circ(y)$. By straightforward calculations we can verify that $|\mathcal{R}^\circ(y)|^2 = \mathcal{R}^\circ(r)$. □

Proposition 9. *Assume the following splitting of the data of the vectorial Ribaucour transformation of the circular lattice* y*:*

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{Y}_C \\ \boldsymbol{r}_C \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}_{C(1)} & \boldsymbol{Y}_{C(2)} \\ \boldsymbol{r}_{C(1)} & \boldsymbol{r}_{C(2)} \end{pmatrix}, \qquad \boldsymbol{\Psi}_C = \begin{pmatrix} \boldsymbol{\Psi}^{(1)}_{C(1)} & \boldsymbol{\Psi}^{(1)}_{C(2)} \\ \boldsymbol{\Psi}^{(2)}_{C(1)} & \boldsymbol{\Psi}^{(2)}_{C(2)} \end{pmatrix},$$

associated with partition $K = K_1 + K_2$*. Then the vectorial Ribaucour transformation* $\mathcal{R}^\circ(y)$ *is equivalent to the following superposition of vectorial Ribaucour transformations:*

1. *Transformation* $\mathcal{R}^\circ_{(1)}(y)$ *with the data* $\boldsymbol{Y}_{C(1)}, \boldsymbol{r}_{C(1)}, \psi^{(1)}, \boldsymbol{\Psi}^{(1)}_{C(1)}$.
2. *Application on the result obtained in point* 1, *transformation* $\mathcal{R}^\circ_{(2)}$ *with the data* $\mathcal{R}^\circ_{(1)}(\boldsymbol{Y}_{C(2)}), \mathcal{R}^\circ_{(1)}(\boldsymbol{r}_{C(2)}), \mathcal{R}^\circ_{(1)}(\psi^{(2)}), \mathcal{R}^\circ_{(1)}(\boldsymbol{\Psi}^{(2)}_{C(2)})$.

Proof. The reasoning is similar to that of the proof of Proposition 5. The only new ingredient is that the vector

$$\mathcal{R}^\circ_{(1)}(\boldsymbol{r}_{C(2)}) = \boldsymbol{r}_{C(2)} - \boldsymbol{r}_{C(1)}(\boldsymbol{\Psi}^{(1)}_{C(1)})^{-1}\boldsymbol{\Psi}^{(1)}_{C(2)}$$

consists of the Combescure transforms of the function

$$\mathcal{R}^\circ_{(1)}(r) = r - \boldsymbol{r}_{C(1)}(\boldsymbol{\Psi}^{(1)}_{C(1)})^{-1}\psi^{(1)} = |\mathcal{R}^\circ_{(1)}(y)|^2. \qquad \square$$

5. Conclusion and final remarks

In this paper we presented the theory of quadrilateral lattices subjected to quadratic reductions. We concentrated our research on the geometric aspect of the problem of quadratic reductions, i.e., our considerations concerned the lattice points, not the corresponding reduction of the MQL equation (2). However, it is worth of mentioning that in [25] it was shown that the circular lattices, for $N = M = 3$, can be described by the discrete BKP equation [10].

The (vectorial) Ribaucour-type transformations of the quadrilateral lattices in quadrics, also constructed in the paper, allow to find new lattices from given ones. In particular, a lot of intersecting examples can be constructed just applying the Ribaucour transformations to the trivial background lattices (see, for example [28,29]). Moreover, one may expect that suitable modification of the scheme, based on the $\bar\partial$- dressing method, applied in [17] to study circular lattices, can be used to study the quadratic reductions as well.

We conclude the paper with a few general remarks on integrable lattices. The multidimensional quadrilateral lattice seems to be quite general integrable lattice and other integrable lattices come as their reductions. Notice [13,16] that the quadrilateral lattices naturally "live" in the projective space. To obtain reductions of the quadrilateral lattice one can follow the Cayley and Klein approach to subgeometries of the projective geometry, which was successfuly applied in [11,12] to the (continuous) Toda systems. The results of the present paper can be considered as the basic tool to construct integrable lattices in spaces obtained by intersection of quadrics. As a particular example, we demonstrated here the close connection of the circular lattices and the Möbius geometry.

Another way to obtain the integrable reductions of the quadrilateral lattices (and the corresponding reductions of Eq. (2)) can be achieved by imposing on the lattice special symmetry conditions. These additional requirements may allow for dimensional reduction of the geometric integrability scheme (see examples and discussion in [14]). In particular, the discrete isothermic surfaces [5], or even the discrete analogs of the holomorphic functions (see, for example [33] and references therein), can be considered as further reductions of the circular lattices.

The third way, pointed out in [14], to obtain new examples of integrable lattices may be to consider quadrilateral lattices (and their reductions) in spaces over fields different from the field of real numbers. In particular, geometries over Galois fields (finite geometries) should give rise to integrable ultradiscrete systems (integrable cellular automata).

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Note added in proof

After submitting the paper I became aware that Lin and Mañas have found independently [38] the vectorial Ribaucour transformation of multidimensional circular lattices and its permutability property.

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